Aufgabe 4.1

Nach dem Residuensatz (Cauchyscher Integralsatz) gilt, da die Funktion holomorph auf $\mathbb{C}$ ist:

$$\begin{aligned}
0 &= \int_{-R}^{R} e^{-(\xi+\beta)^2} d\xi + \int_{R}^{-R} e^{-(\xi-i\Im(\beta)+\beta)^2} d\xi + \int_{0}^{-\Im\beta} e^{-(R+iz+\beta)^2} dz + \int_{-\Im\beta}^{0} e^{-(-R+iz+\beta)^2} dz \\
&= \int_{-R}^{R} e^{-(\xi+\beta)^2} d\xi - \int_{-R}^{R} e^{-(\xi+\Re(\beta))^2} d\xi + \int_{0}^{-\Im\beta} e^{-(R+iz+\beta)^2} - e^{-(-R+iz+\beta)^2} dz \\
&= \int_{-R}^{R} e^{-(\xi+\beta)^2} d\xi - \int_{-R+\Re(\beta)}^{R+\Re(\beta)} e^{-(\xi)^2} d\xi + \\
&\int_{0}^{-\Im\beta} \left[e^{-(R^2-z^2+\beta^2+2Riz+2R\beta+2\beta iz)} - e^{-(R^2-z^2+\beta^2-2Riz-2R\beta+2\beta iz)} \right] dz \\
&= \int_{-R}^{R} e^{-(\xi+\beta)^2} d\xi - \int_{-R+\Re(\beta)}^{R+\Re(\beta)} e^{-(\xi)^2} d\xi + \\
&\int_{0}^{-\Im\beta} \left[e^{-R^2+z^2-\beta^2-2R\beta}\; e^{-2Riz-2\beta iz} - e^{-R^2+z^2-\beta^2+2R\beta}\; e^{+2Riz-2\beta iz} \right] dz
\end{aligned}$$

Im Grenzfall $\lim\limits_{R\to\infty}$ gilt:

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-(\xi+\beta)^2} d\xi &= \underbrace{\lim_{R\to\infty} \int_{-R+\Re(\beta)}^{R+\Re(\beta)} e^{-(\xi)^2} d\xi}_{=\sqrt{\pi}} + \\
&\lim_{R\to\infty} \int_{0}^{-\Im\beta} \left[e^{-R^2+z^2-\beta^2+2R\beta}\; e^{+2Riz-2\beta iz} - e^{-R^2+z^2-\beta^2-2R\beta}\; e^{-2Riz-2\beta iz} \right] dz \\
&= \sqrt{\pi} + \lim_{R\to\infty} \int_{0}^{-\Im\beta} \left[\underbrace{e^{-R^2+z^2-\beta^2+2R\beta}}_{\to 0}\; \underbrace{e^{+2Riz-2\beta iz}}_{\text{beschränkt}} - \underbrace{e^{-R^2+z^2-\beta^2-2R\beta}}_{\to 0}\; \underbrace{e^{-2Riz-2\beta iz}}_{\text{beschränkt}} \right] dz \\
&= \sqrt{\pi} + 0 = \int_{-\infty}^{\infty} e^{-(\xi)^2} d\xi
\end{aligned}$$

Aufgabe 4.2

$$\begin{aligned}
\left(\int_{-\infty}^{+\infty} e^{-\xi^2} d\xi \right)^2 &= \left(\int_{-\infty}^{+\infty} e^{-x^2} dx \right) \cdot \left(\int_{-\infty}^{+\infty} e^{-y^2} dy \right) \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\; dx\; dy \\
&= \int_{0}^{+\infty} \int_{0}^{2\pi} e^{-\rho^2}\; \rho\; d\phi\; d\rho \\
&= \int_{0}^{+\infty} 2\pi\; e^{-\rho^2}\; \rho\; d\rho = 2\;\pi \left[-\frac{1}{2} e^{-\rho^2} \right]_{\rho=0}^{+\infty} = \pi
\end{aligned}$$

$$\Rightarrow \int_{-\infty}^{+\infty} e^{-\xi^2} d\xi = \sqrt{\pi}$$

Aufgabe 4.3

$$\begin{aligned}
0 &= \int_{R}^{0} e^{-(\xi)^2} d\xi + \int_{0}^{R} e^{-(e^{i\phi}\xi)^2} d\xi + \int_{\phi}^{0} e^{-(e^{i\phi}R)^2} d\phi \\
&= \int_{R}^{0} e^{-(\xi)^2} d\xi + \int_{0}^{R} e^{-(e^{i\phi}\xi)^2} d\xi + \int_{\phi}^{0} e^{-R^2(1-2\sin^2\phi)} e^{-R^2(2i\sin\phi\cos\phi)} d\phi
\end{aligned}$$

Im Grenzfall $\lim_{R\to\infty}$ gilt:

$$
\begin{aligned}
\int_0^\infty e^{-(e^{i\phi}\xi)^2} d\xi &= \underbrace{-\int_R^0 e^{-(\xi)^2} d\xi}_{=\frac{1}{2}\sqrt{\pi}} - \int_\phi^0 \underbrace{e^{-R^2(1-2\sin^2\phi)}}_{=0 \text{ für } \phi\leq 45^\circ} \underbrace{e^{-R^2(i\sin 2\phi)}}_{\text{beschränkt}} d\phi \\
&= \frac{1}{2}\sqrt{\pi} \\
&\Rightarrow \int_0^\infty e^{-(e^{i\phi}\xi)^2} d\xi = \sqrt{\pi}
\end{aligned}
$$

$$
\int_{-\infty}^\infty e^{-|\alpha|^2(e^{i\phi}\xi)^2} d\xi = \frac{1}{|\alpha|}\int_{-\infty}^\infty e^{-|\alpha|^2(e^{i\phi}\frac{x}{|\alpha|})^2} dx = \frac{\sqrt{\pi}}{|\alpha|}
$$

Daraus folgt Behauptung.

Aufgabe 4.4

$$
\begin{aligned}
\psi(x,t=0) &= c\int_{-\infty}^{+\infty} \exp\left(-\left(\frac{a}{2}\right)^2 (k-k_0)^2\right)\cdot\exp(ikx)\,dk \\
&= c\int_{-\infty}^{+\infty} \exp\left(-\left(\frac{a}{2}\right)^2 (k^2-2kk_0+k_0^2)+ikx\right) dk \qquad \text{sei } b=\left(\frac{a}{2}\right)^2 \\
&= c\int_{-\infty}^{+\infty} \exp\left(-b\left(k^2-2kk_0+k_0^2-\frac{ikx}{b}\right)\right) dk \\
&= c\int_{-\infty}^{+\infty} \exp\left(-b\left(k^2-2k\left(k_0+\frac{ix}{2b}\right)+k_0^2\right)\right) dk \\
&= c\int_{-\infty}^{+\infty} \exp\left(-b\left(k-\left(k_0+\frac{ix}{2b}\right)\right)^2 - bk_0^2 + b\left(k_0+\frac{ix}{2b}\right)^2\right) dk \\
&= c\int_{-\infty}^{+\infty} \exp\left(-bk^2-bk_0^2+bk_0^2+ixk_0-\frac{x^2}{4b}\right) dk \\
&= c\,\exp\left(ixk_0-\frac{x^2}{4b}\right)\int_{-\infty}^{+\infty} \exp\left(-bk^2\right) dk \\
&= 2\frac{c}{a}\sqrt{\pi}\,\exp\left(ixk_0-\frac{x^2}{a^2}\right)
\end{aligned}
$$

Aufgabe 5

a) Aus den Randbedingungen $x_{cl}(0) = 0$ und $x_{cl}(T) = L$ folgt:

$$\begin{aligned} x_{cl}(t) &= \frac{L}{T} \cdot t \\ \dot{x}_{cl}(t) &= \frac{L}{T} \end{aligned}$$

b) Das Wirkungsintegral ist:

$$\begin{aligned} S &= \int_0^T \frac{m}{2} \dot{x}_{cl}^2(t) dt \\ &= \int_0^T \frac{m\ L^2}{2\ T^2} t^2 dt = \frac{1}{6} m\ L^2\ T \end{aligned}$$

c) Es muss gelten:

$$x_0 = 0$$

$$vT + gT^2 = L$$

d) Das Wirkungsintegral in Abhängigkeit von v ist:

$$\begin{aligned} S(v) &= \int_0^T \frac{m}{2} \dot{x}^2(t) dt \\ &= \int_0^T \frac{m}{2} (v + 2gt)^2(t) dt \\ &= \frac{m}{2} \left(v^2 T + 2vgT^2 + \frac{4}{3} g^2 T^3 \right) \\ &= \frac{m}{2} \left(v^2 T + 2v(L - vT) + \frac{4}{3} T \left(\frac{L}{T} - v \right)^2 \right) \\ &= \frac{m}{2} \left(v^2 T + 2vL - 2v^2 T + \frac{4}{3} \frac{L^2}{T} - \frac{8}{3} Lv + \frac{4}{3} T v^2 \right) \\ &= \frac{m}{6} \left(v^2 T - 2vL + 4 \frac{L^2}{T} \right) \\ &= \frac{Tm}{6} \left(\left(v - \frac{L}{T} \right)^2 + c \right) \\ & \Rightarrow \text{minimal für } v = v_{cl} \ \text{ bzw. } \ g = 0 \end{aligned}$$

$$\begin{aligned} \Delta S &= S(v_{cl}) - S(v_{cl} - \Delta v) \\ &= \frac{1}{6} T m \Delta v^2 \end{aligned}$$

e) Die maximale Abweichung vom Weg ist:

$$\begin{aligned} \max \Delta x &= \max \left(x_{cl} - x_\hbar \right) \\ &= \max \left(v_{cl} t - (v_{cl} + \Delta v) t - g t^2) \right) \\ &= \max \left(-\Delta v t - \frac{L - (v_{cl} + \Delta v) T}{T^2} t^2 \right) \\ &= \Delta v \cdot \max \left(-t - \frac{1}{T} t^2 \right) \\ &= \Delta v \cdot \max \left(- \left(\frac{t}{\sqrt{T}} + \frac{\sqrt{T}}{2} \right)^2 + \frac{1}{4} T \right) \\ &= \frac{1}{4} \Delta v T = \sqrt{\frac{3}{8} \frac{\hbar T}{m}} \end{aligned}$$